

Constructing positive maps from block matrices

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Abstract

Positive maps are useful for detecting entanglement in quantum information theory. Any entangled state can be detected by some positive map. In this paper, the relation between positive block matrices and completely positive trace-preserving maps is characterized, from which a new method for constructing decomposable maps is derived. In addition, a method for constructing positive maps from Hermitian block matrices is also obtained.

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1. Introduction

Positive map plays a crucial role in quantum information theory [1, 2, 3, 4]. In the context of quantum physics, every quantum operation is characterized by a trace-preserving completely positive map transforming quantum states to quantum states, where a quantum state is described by a density matrix, i.e., a positive matrix with unit trace. Positive but not completely positive (PNCP) maps are useful tools for detecting entanglement of quantum states.

Recall that a bipartite quantum state $\rho \in \mathcal{M}_m \otimes \mathcal{M}_n$ is called *separable* if it can be written as a convex combination [5, 6]

$$\rho = \sum_i p_i \rho_i^{(1)} \otimes \rho_i^{(2)}, \quad \sum_i p_i = 1, \quad p_i \geq 0, \quad (1)$$

where $\rho_i^{(1)}$ and $\rho_i^{(2)}$ are quantum states in \mathcal{M}_m (i.e., the algebra of all $m \times m$ complex matrices) and \mathcal{M}_n , respectively. Otherwise, ρ is called *entangled*. It is well known that a bipartite state $\rho \in \mathcal{M}_m \otimes \mathcal{M}_n$ is separable if and only if [2]

$$(\text{id}_m \otimes \Phi)\rho \geq 0 \quad (2)$$

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holds for any PNCP map $\Phi : \mathcal{M}_m \rightarrow \mathcal{M}_n$.

A well-known example of PNCP map is the transpose τ , another one is the so-called reduction map defined by $\Phi(A) = \text{tr}(A)I - A$ [7]. The trace map $\text{tr}(\cdot)$ is one of the most frequently-used completely positive (CP) maps in quantum literature. Every CP map $\Phi : \mathcal{M}_m \rightarrow \mathcal{M}_n$ admits the form of [9]

$$\Phi(A) = \sum_i X_i A X_i^\dagger, \quad A \in \mathcal{M}_m, \quad (3)$$

where X_i are $n \times m$ matrices. In particular, Φ is trace-preserving (i.e., $\text{tr}(\Phi(A)) = \text{tr}(A)$ for any $A \in \mathcal{M}_m$) if and only if $\sum_i X_i^\dagger X_i = I_m$, and in such a case, it is called a *quantum channel* in quantum physics.

The structure of positive maps has been studied extensively by researchers on both mathematics and quantum physics [8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19]. The main purpose of this article is to propose a new method for constructing positive maps, from which we get some useful tools for detecting entanglement in quantum information theory.

In Sec. 2 we characterize the relation between the positive block matrices and the trace-preserving PNCP maps (Lemma 2.1), and reveal some relation between bipartite states and quantum channels (Corollary 2.2). Several special bipartite states are considered, which corresponds to special quantum channels. In addition, we obtain a sufficient condition for a bipartite state to be separable (Corollary 2.3). In Sec. 3, the correspondence between decomposable PNCP maps and NPPT (non positive partial transpose) positive block matrices is derived (Theorem 3.1). It is illustrated with several examples of new decomposable PNCP maps. Sec. 4 is devoted to a new way of constructing PNCP maps from Hermitian block matrices (Theorem 4.1).

For clarity, we list the notations and the terminologies in this note. A^t denotes the transpose of $A \in \mathcal{M}_m$, and A^\dagger stands for the transpose of the complex conjugate of A , i.e., $A^\dagger = \bar{A}^t$, A is positive (or positive semi-definite), denoted by $A \geq 0$, if $A^\dagger = A$ and its eigenvalues are nonnegative. Let \mathcal{M}_m^+ be the positive part of \mathcal{M}_m . $\mathcal{M}_m \otimes \mathcal{M}_n (= \mathcal{M}_m(\mathcal{M}_n))$ is the algebra of all $m \times m$ block matrices with $n \times n$ complex matrices as entries. A linear map $\Phi : \mathcal{M}_m \rightarrow \mathcal{M}_n$ is *positive* (resp. *Hermitian*) if $\Phi(\mathcal{M}_m^+) \subseteq \mathcal{M}_n^+$ (resp. $\Phi(A)$ is Hermitian for any Hermitian matrix $A \in \mathcal{M}_m$). Let E_{ij} be the matrix units of the associated matrix algebra. Φ is *completely positive* if $(\text{id}_k \otimes \Phi)(\sum_{i,j=1}^k E_{ij} \otimes A_{ij}) = \sum_{i,j=1}^k E_{ij} \otimes \Phi(A_{ij})$ is positive for any positive integer k and every $A = \sum_{i,j=1}^k E_{ij} \otimes A_{ij} \in (\mathcal{M}_k \otimes \mathcal{M}_m)^+$. A positive map Φ is *decomposable* if there exist two CP maps Φ_1 and Φ_2 such that $\Phi = \Phi_1 + \Phi_2 \circ \tau$, where τ denotes the transpose map (i.e., $\tau(A) = A^t$, $A \in \mathcal{M}_m$). Otherwise, Φ is defined to be *indecomposable*. For $A = \sum_{i,j=1}^m E_{ij} \otimes A_{ij} \in \mathcal{M}_m \otimes \mathcal{M}_n$, $A^{t_1} = \sum_{i,j=1}^m E_{ij}^t \otimes A_{ij}$ and $A^{t_2} = \sum_{i,j=1}^m E_{ij} \otimes A_{ij}^t$ are called the partial transpose of A . It is clear that $(A^{t_1})^t = A^{t_2}$, $(A^{t_2})^t = A^{t_1}$ and $(A^{t_1})^{t_2} = A^t$. We call A a positive partial transpose (PPT) matrix if $A^{t_{1,2}} \geq 0$.

For convenience in the quantum physics sense, we use the Dirac notation in the following. A ket $|\psi\rangle$ denotes a unit column vector in \mathbb{C}^m and the bra $\langle\psi|$

is a row vector whose entries are the complex conjugates of those of $|\psi\rangle$. For vectors $|\psi\rangle \in \mathbb{C}^m$ and $|\phi\rangle \in \mathbb{C}^n$, the out product $|\psi\rangle\langle\phi|$ is a rank-one complex matrix of size $m \times n$ projecting along $|\psi\rangle$, and $|\psi\rangle|\phi\rangle = |\psi\rangle \otimes |\phi\rangle$ which is a column vector in $\mathbb{C}^m \otimes \mathbb{C}^n = \mathbb{C}^{mn}$.

2. CP maps & positive block matrices

For $A = \sum_{i,j=1}^m E_{ij} \otimes A_{ij} \in \mathcal{M}_m \otimes \mathcal{M}_n$, the *reduced matrix* of A , denoted by $A_{1,2}$, is defined by

$$\begin{aligned} A_1 = \text{tr}_2(A) &:= (\text{id}_m \otimes \text{tr}) \sum_{i,j=1}^m E_{ij} \otimes A_{ij} = \sum_{i,j=1}^m \text{tr}(A_{ij}) E_{ij}, \\ A_2 = \text{tr}_1(A) &:= (\text{tr} \otimes \text{id}_n) \sum_{i,j=1}^m E_{ij} \otimes A_{ij} = \sum_{i,j=1}^m \text{tr}(E_{ij}) A_{ij} = \sum_i A_{ii}, \end{aligned} \quad (4)$$

where $\text{tr}(\cdot)$ denotes the trace operation. It is obvious that $\text{tr}(A) = \text{tr}(A_1) = \text{tr}(A_2)$ and $A_{1,2} \geq 0$ whenever $A \geq 0$. We start our discussing with the famous *Schmidt decomposition* theorem which reads as: Let H_1 and H_2 be two complex Hilbert spaces with $\dim H_1 = m$ and $\dim H_2 = n$, and let $|x\rangle$ be a vector (not necessarily normalized) in $H_1 \otimes H_2$, then there exist orthonormal sets $\{|e_i\rangle\}$ and $\{|f_i\rangle\}$ of H_1 and H_2 respectively, and positive numbers $\{\lambda_i\}$, such that

$$|x\rangle = \sum_i^{r(x)} \lambda_i |e_i\rangle |f_i\rangle, \quad (5)$$

where $\sum_i \lambda_i^2 = \| |x\rangle \|^2$, $\{\lambda_i\}$ are called the Schmidt coefficients of $|x\rangle$ and $r(x) \leq \min\{m, n\}$ is the Schmidt number of $|x\rangle$.

The following lemma is necessary.

Lemma 2.1 *Let $A \in (\mathcal{M}_m \otimes \mathcal{M}_n)^+$ and $|x\rangle \in \mathbb{C}^m \otimes \mathbb{C}^n$ be a vector (not necessarily normalized) with $\text{tr}_2(|x\rangle\langle x|) = A_1$, then there exists a trace-preserving CP map $\Lambda : \mathcal{M}_m \rightarrow \mathcal{M}_n$ such that*

$$A = (\text{id}_m \otimes \Lambda) |x\rangle\langle x|. \quad (6)$$

Proof Let $A = \sum_{i,j=1}^m E_{ij} \otimes A_{ij}$, then $A_1 = \sum_{i,j=1}^m \text{tr}(A_{ij}) E_{ij}$. Let

$$A_1 = \sum_i \lambda_i^2 |\psi_i\rangle\langle\psi_i|$$

be the spectral decomposition of A_1 . Suppose

$$|x\rangle = \sum_{i=1}^{r(x)} \lambda_i |\psi_i\rangle |\phi_i\rangle$$

for some orthonormal set $\{|\phi_i\rangle\}$ in \mathbb{C}^m and let $\{|\psi_i\rangle\}_{i=1}^m$ be an orthonormal basis of \mathbb{C}^m induced from the eigenvectors of $A_1 - \{|\psi_i\rangle\}_{i=1}^r$ (we assume with no loss of generality that $\text{rank}(A_1) = r(x) = r$, $1 \leq r < m$; the case of $r = m$ can be discussed similarly). We write $E'_{ij} = |\psi_i\rangle\langle\psi_j|$ and $E_{ij} = \sum_{k,l=1}^m \omega_{kl}^{(ij)} E'_{kl}$ for some complex numbers $\{\omega_{kl}^{(ij)}\}$, then

$$\begin{aligned} A &= \sum_{i,j=1}^m \left(\sum_{k,l=1}^m \omega_{kl}^{(ij)} E'_{kl} \right) \otimes A_{ij} \\ &= \sum_{k,l=1}^m E'_{kl} \otimes \left(\sum_{i,j=1}^m \omega_{kl}^{(ij)} A_{ij} \right) = \sum_{i,j=1}^m E'_{ij} \otimes A'_{ij}, \end{aligned}$$

where $A'_{ij} = \sum_{k,l=1}^m \omega_{kl}^{(ij)} A_{kl}$. It follows from

$$A_1 = \sum_{i,j=1}^m \text{tr}(A'_{ij}) E'_{ij} = \sum_{i=1}^r \lambda_i^2 E'_{ii}$$

that

$$\text{tr}(A'_{ij}) = \begin{cases} 0 & \text{when } i \neq j, \\ \lambda_i^2 & \text{when } i = j, \end{cases}$$

and

$$\text{tr}(A'_{ij}) = 0 \quad \text{when } i > r \text{ or } j > r.$$

Since $\text{tr}(A'_{ii}) = 0$ implies $A'_{ii} = 0$ when $i > r$, and thus $A'_{ij} = 0$ when $i > r$ or $j > r$. Consequently,

$$A = \sum_{i,j=1}^r E'_{ij} \otimes A'_{ij}.$$

Let $\{|\phi_i\rangle\}_{i=1}^n$ be an orthonormal basis of \mathbb{C}^n extended from $\{|\phi_i\rangle\}_{i=1}^r$. We now define a linear map $\Lambda : \mathcal{M}_m \rightarrow \mathcal{M}_n$ by

$$\Lambda(|\phi_i\rangle\langle\phi_j|) = A''_{ij} = \begin{cases} \frac{1}{\lambda_i \lambda_j} A'_{ij} & \text{when } 1 \leq i, j \leq r, \\ X_i & \text{when } r < i = j \leq n, \\ 0 & \text{when } r < i, j \leq n, i \neq j \end{cases} \quad (7)$$

where X_i are positive matrices in \mathcal{M}_n with $\text{tr}(X_i) = 1$, $r \leq i \leq n$. then $A = (\text{id}_m \otimes \Lambda)|x\rangle\langle x|$. It remains to show that Λ is a trace-preserving CP map. By the definition above, it is easy to see that Λ preserves the trace. We next show that it is completely positive. For clarity, we denote by A' and A'' the block matrices $\sum_{i,j=1}^r E'_{ij} \otimes A'_{ij} + \sum_{i>r} E'_{ii} \otimes X_i$ and $\sum_{i,j=1}^m E'_{ij} \otimes A''_{ij}$ respectively

under the basis $\{E'_{ij}\}$ of \mathcal{M}_r . Observe that

$$= \begin{bmatrix} \frac{1}{\lambda_1} I & & & & \\ & \ddots & & & \\ & & \frac{1}{\lambda_r} I & & \\ & & & I & \\ & & & & \ddots \\ & & & & & I \end{bmatrix} A' \begin{bmatrix} \frac{1}{\lambda_1} I & & & & \\ & \ddots & & & \\ & & \frac{1}{\lambda_r} I & & \\ & & & I & \\ & & & & \ddots \\ & & & & & I \end{bmatrix},$$

where I is the $n \times n$ unit matrix. Therefore $A'' \geq 0$ iff $A' \geq 0$, and in turn, iff $A \geq 0$, which implies that Λ is a CP map since the positive block matrix A'' is the Choi matrix of Λ [9]. \square

The lemma above implies that any positive block matrix can induce trace-preserving CP maps.

In general, let A be a Hermitian matrix in $\mathcal{M}_m \otimes \mathcal{M}_n$ and $|x\rangle \in \mathbb{C}^m \otimes \mathbb{C}^m$ be a vector (not necessarily normalized) with $A_1 = \text{tr}_2(|x\rangle\langle x|)$, then there exists a trace-preserving Hermitian map $\Lambda : \mathcal{M}_m \rightarrow \mathcal{M}_n$ such that $A = (\text{id}_m \otimes \Lambda)|x\rangle\langle x|$. (Notice that Λ is a Hermitian map if and only if the Choi matrix of Λ is Hermitian [9].)

Remark i) The trace-preserving CP map satisfying Eq. (6) is not unique. If $|x\rangle = \sum_{i=1}^{r(x)} \lambda_i |\psi_i\rangle |\psi_i\rangle \in \mathbb{C}^m \otimes \mathbb{C}^m$ is its Schmidt decomposition such that $A_1 = \text{tr}_2(|x\rangle\langle x|)$ and $r(x) = m$, then the trace-preserving CP map Λ is unique. ii) The Eq. (6) is different from the Choi-Jamiolkowski isomorphism between block matrices and positive maps. Recall that the Choi-Jamiolkowski isomorphism reads as [9, 11]

$$W = (\text{id}_m \otimes \Phi) \left(\sum_{i,j=1}^m E_{ij} \otimes E_{ij} \right) = \sum_{i,j=1}^m E_{ij} \otimes \Phi(E_{ij}), \quad (8)$$

where $W \in \mathcal{M}_m \otimes \mathcal{M}_n$ and Φ is a positive map from \mathcal{M}_m to \mathcal{M}_n . The Choi matrix W is uniquely determined by Φ and Φ is unique when W is fixed. In addition, the map Φ in Eq. (8) is not necessarily trace-preserving. iii) If $\text{rank}(A_1) = r < m$, then A can be viewed as a matrix in $\mathcal{M}_r \otimes \mathcal{M}_n$.

In quantum literature the term *pure state* is sometimes used for both rank-one density matrix $|\psi\rangle\langle\psi|$ and the ket $|\psi\rangle$. A pure state $|x\rangle$ is called a *purification* of a density matrix ρ if $\rho_1 = \text{tr}_2(|x\rangle\langle x|)$.

The following are some consequences of Lemma 2.1.

Corollary 2.2 *Let $\rho \in \mathcal{M}_m \otimes \mathcal{M}_n$ be a bipartite density matrix and $|x\rangle \in \mathbb{C}^m \otimes \mathbb{C}^m$ be a purification of the reduced density matrix ρ_1 . Then there exists a quantum channel Λ from \mathcal{M}_m to \mathcal{M}_n such that*

$$\rho = (\text{id}_m \otimes \Lambda)|x\rangle\langle x|. \quad (9)$$

That is, any bipartite state arises from a quantum channel acting on the purification of the reduced state.

Let $\rho \in \mathcal{M}_m \otimes \mathcal{M}_n$ be a bipartite density matrix and ρ_1 be a pure state. By Corollary 2.2, ρ is separable, in particular, ρ is a product state, i.e., $\rho = \rho_1 \otimes \rho_2$.

Corollary 2.3 *Let $\rho \in \mathcal{M}_m \otimes \mathcal{M}_m$ be a bipartite density matrix. If either $m = 2$ and $\text{rank}(\rho_1) \leq 3$ or $m = 3$ and $\text{rank}(\rho_1) \leq 2$, then ρ is separable iff it is PPT.*

Proof If $m = 2$ (or $m = 3$) and $\text{rank}(\rho_1) = r \leq 3$ (or $r \leq 2$), then $\rho = \sum_{i=1}^r E'_{ij} \otimes A'_{ij}$ is in fact a state in $r \otimes 2$ (or $r \otimes 3$) bipartite system. The theorem is now clear from the fact that a state in $m \otimes n$ system with $mn \leq 6$ is separable iff it is PPT [2]. \square

At the end of this section, we list some special bipartite states which lead to special quantum channels. A quantum channel $\Lambda : \mathcal{M}_m \rightarrow \mathcal{M}_n$ is *entanglement breaking* if $(\text{id}_k \otimes \Lambda)\rho$ is separable for any state $\rho \in \mathcal{M}_k \otimes \mathcal{M}_m$ [20]. It is showed in [20] that Λ is entanglement breaking iff $\Lambda(A) = \sum_k \text{tr}(W_k A) \varrho_k$ for any $A \in \mathcal{M}_m$, where each ϱ_k is a density matrix in \mathcal{M}_n , $W_k \geq 0$ and $\sum_k W_k = I_m$. Λ is called a *completely contractive* channel if $\Lambda(A) = \text{tr}(A)\sigma$ holds for any $A \in \mathcal{M}_m$ for some fixed state $\sigma \in \mathcal{M}_n$ [21]. Let ρ be a bipartite density matrix in $\mathcal{M}_m \otimes \mathcal{M}_n$. If ρ is a classical-quantum state (i.e., $\rho = \sum_i p_i |i\rangle\langle i| \otimes \sigma_i$ with $\{|i\rangle\}$ a orthonormal set of \mathbb{C}^m and σ_i are density matrices in \mathcal{M}_n) and $\text{rank}(\rho_1) = m$, then one can check that the channel Λ in Eq. (9) is entanglement breaking. In particular, if ρ is a product state with $\text{rank}(\rho_1) = m$, then the channel Λ in Eq. (9) is completely contractive. If $\rho = |y\rangle\langle y|$ is a pure state in $\mathcal{M}_m \otimes \mathcal{M}_m$ with $r(y) = m$, then Λ in Eq. (9) is a unitary operation, i.e., $\Lambda(A) = UAU^\dagger$ for some unitary matrix in \mathcal{M}_m .

3. Decomposable maps derived from NPPT positive block matrices

For simplicity, we fix some notations. Let A be a positive matrix in $\mathcal{M}_m \otimes \mathcal{M}_n$. Write

$$A = \sum_{i,j=1}^m E_{ij} \otimes A_{ij} = \sum_{k,l=1}^n \tilde{A}_{kl} \otimes E_{kl}, \quad (10)$$

where E_{ij} are matrix units in \mathcal{M}_m and E_{kl} are matrix units in \mathcal{M}_n . That is, A can be denoted by $[A_{ij}]$ or $[\tilde{A}_{kl}]$. It is straightforward that

$$\tilde{A}_{kl} = [a_{ij}^{(kl)}] \text{ iff } A_{ij} = [a_{kl}^{(ij)}].$$

Let $A_1 = \text{tr}_2(A) = \sum_{i=1}^r \lambda_i |\psi_i\rangle\langle\psi_i|$ be its spectral decomposition and $E'_{ij} = |\psi_i\rangle\langle\psi_j|$. Then A can be represented as $A = \sum_{i,j=1}^r E'_{ij} \otimes A'_{ij}$, where A'_{ij} are $n \times n$ matrices. Let A''_{ij} defined as in Eq. (7), it is clear that $A^{t_2} \geq 0$ iff $(A'')^{t_2} \geq 0$, $A'' = [A''_{ij}] = \sum_{i,j=1}^m E'_{ij} \otimes A''_{ij}$.

Theorem 3.1 *Let A be a positive matrix in $\mathcal{M}_m \otimes \mathcal{M}_n$ and $A^{t_2} \not\geq 0$. If $\Phi : \mathcal{M}_m \rightarrow \mathcal{M}_n$ is a linear map satisfying $\Phi(E'_{ij}) = (A''_{ij})^t$, where E'_{ij}, A''_{ij} are defined as above, then Φ is a PNCP map, moreover, it is trace-preserving and decomposable.*

Proof Since the Choi matrix of Φ , i.e., $(\text{id}_m \otimes \Phi)(\sum_{i,j=1}^m E_{ij} \otimes E'_{ij}) = (A'')^{t_2}$, is not positive, by Theorem 2 in [9], Φ is not completely positive. In order to show the positivity of Φ , it suffices to prove

$$\Phi(|w\rangle\langle w|) \geq 0$$

holds for any rank-one projection $|w\rangle\langle w| \in \mathcal{M}_m$. Let $|w\rangle\langle w| = \sum_{i,j=1}^m t_{ij} E'_{ij}$. We claim that $T = [t_{ij}]$ is a rank-one projection. In fact, there exists a unitary matrix U such that $U|\psi_i\rangle = |e_i\rangle$, $i = 1, 2, \dots, m$, where $\{|\psi_i\rangle\}$ is the orthonormal basis of \mathbb{C}^m derived from the eigenvectors of A_1 , and $\{|e_i\rangle\}$ is the standard orthonormal basis of \mathbb{C}^m . It turns out that

$$U|w\rangle\langle w|U^\dagger = \sum_{i,j=1}^m t_{ij} U E'_{ij} U^\dagger = \sum_{i,j=1}^m t_{ij} E_{ij} = [t_{ij}],$$

which implies that T is a rank-one projection. Define $\Lambda : \mathcal{M}_m \rightarrow \mathcal{M}_n$ by $\Lambda(E'_{ij}) = A''_{ij}$. By Lemma 2.1, it is completely positive, and thus positive. Hence

$$\Lambda(|w\rangle\langle w|) = \sum_{i,j=1}^m t_{ij} \Lambda(E'_{ij}) = \sum_{i,j=1}^m t_{ij} A''_{ij} = [\text{tr}(T^t \tilde{A}''_{ij})] \geq 0. \quad (11)$$

It follows that

$$\begin{aligned} \Phi(|w\rangle\langle w|) &= \sum_{i,j=1}^m t_{ij} \Phi(E'_{ij}) = \sum_{i,j=1}^m t_{ij} (A''_{ij})^t \\ &= [\text{tr}(T^t (\tilde{A}''_{ij})^t)] = [\text{tr}(T \tilde{A}''_{ij})] = \Lambda(|\bar{w}\rangle\langle \bar{w}|) \geq 0, \end{aligned}$$

that is Φ is positive. Define $\Lambda' : \mathcal{M}_m \rightarrow \mathcal{M}_n$ by $\Lambda'(X) = \overline{\Lambda(X)}$, where $\overline{\Lambda(X)}$ denotes the complex conjugate of $\Lambda(X)$. Then Λ' is completely positive. Since $\Phi = \Lambda' \circ \tau$, Φ is decomposable. It is easy to see that Φ is trace-preserving from the definition. \square

We illustrate our results with some well-known bipartite density matrices.

Example 3.2 We consider a $3 \otimes 3$ density matrix,

$$\rho(a) = \frac{1}{21} \left[\begin{array}{ccc|ccc|ccc} 2 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 2 \\ 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5-a & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 5-a & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5-a & 0 \\ 2 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 2 \end{array} \right],$$

where $2 \leq a \leq 5$. It is proved in [22] that $\rho(a)$ is separable iff $2 \leq a \leq 3$; is PPT entangled iff $3 < a \leq 4$; is non-PPT entangled iff $4 < a \leq 5$. One can check that $\rho(a)$ is non-PPT when $0 \leq a < 1$. By Theorem 3.1,

$$\begin{aligned} & \Phi_1^a([a_{ij}]) \\ = & 2 \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \\ & + \begin{bmatrix} (5-a)a_{22} + aa_{33} & 0 & 0 \\ 0 & aa_{11} + (5-a)a_{33} & 0 \\ 0 & 0 & (5-a)a_{11} + aa_{22} \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} & \Phi_2^a([a_{ij}]) \\ = & 2 \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \\ & + \begin{bmatrix} aa_{22} + (5-a)a_{33} & 0 & 0 \\ 0 & (5-a)a_{11} + aa_{33} & 0 \\ 0 & 0 & aa_{11} + (5-a)a_{22} \end{bmatrix} \end{aligned}$$

are positive but not completely positive and decomposable when $4 < a \leq 5$ or $0 \leq a < 1$. In fact, $\Phi_{1,2}^a$ is positive iff $0 \leq a \leq 5$ and is completely positive iff $2 \leq a \leq 4$.

Example 3.3 We consider the $m \otimes m$ Werner state

$$\omega = \frac{m-x}{m^3-m} I_A \otimes I_B + \frac{mx-1}{m^3-m} F, \quad x \in [-1, 1], \quad (12)$$

with $F = \sum_{i,j=1}^m E_{ij} \otimes E_{ji}$ being the flip operator. It is known that [23] ω is separable iff it is PPT and in turn, iff $0 \leq x \leq 1$. Take $m = 3$, that is

$$\omega = \left[\begin{array}{ccc|ccc|ccc} \frac{1+x}{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{3-x}{24} & 0 & \frac{3x-1}{24} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{3-x}{24} & 0 & 0 & 0 & \frac{3x-1}{24} & 0 & 0 \\ \hline 0 & \frac{3x-1}{24} & 0 & \frac{3-x}{24} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1+x}{12} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{3-x}{24} & 0 & \frac{3x-1}{24} & 0 \\ \hline 0 & 0 & \frac{3x-1}{24} & 0 & 0 & 0 & \frac{3-x}{24} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{3x-1}{24} & 0 & \frac{3-x}{24} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1+x}{12} \end{array} \right].$$

It turns out that

$$\Phi_3^{3,x}(A) = (3x-1)A + (3-x)\text{tr}(A)I, \quad A \in \mathcal{M}_3,$$

is completely positive when $0 \leq x \leq 1$, and is positive but not completely positive and decomposable when $-1 \leq x < 0$. In general,

$$\Phi_3^{m,x}(A) = (mx - 1)A + (m - x)\text{tr}(A)I, \quad A \in \mathcal{M}_m, \quad (13)$$

is positive but not completely positive and decomposable when $-1 \leq x < 0$. Interestingly, for the case of $x = -1$, it is the well-known reduction map. Furthermore, one can check that $\Phi_3^{m,x}$ is completely positive iff $0 \leq x \leq m$ and if positive iff $-1 \leq x \leq m$.

Example 3.4 For the $m \otimes m$ isotropic state

$$\varsigma = \frac{1-y}{m^2}I_m \otimes I_m + yP^+, \quad -\frac{1}{m^2-1} \leq y \leq 1, \quad (14)$$

with $P^+ = \frac{1}{m} \sum_{i,j=1}^m E_{ij} \otimes E_{ij}$ is the so-called maximally entangled state. It is known that ς is separable iff $y \leq \frac{1}{m+1}$, and in turn, iff it is PPT [7]. For the case of $m = 3$,

$$\varsigma = \frac{1}{9} \begin{bmatrix} 2y+1 & 0 & 0 & 0 & 3y & 0 & 0 & 0 & 3y \\ 0 & 1-y & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1-y & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1-y & 0 & 0 & 0 & 0 & 0 \\ 3y & 0 & 0 & 0 & 2y+1 & 0 & 0 & 0 & 3y \\ 0 & 0 & 0 & 0 & 0 & 1-y & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1-y & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1-y & 0 \\ 3y & 0 & 0 & 0 & 0 & 3y & 0 & 0 & 2y+1 \end{bmatrix}.$$

It turns out that

$$\Phi_4^{3,y}(A) = 3yA^t + (1-y)\text{tr}(A)I, \quad A \in \mathcal{M}_3,$$

is positive but not completely positive and decomposable when $\frac{1}{4} < y \leq 1$. In general, we have

$$\Phi_4^{m,y}(A) = myA^t + (1-y)\text{tr}(A)I, \quad A \in \mathcal{M}_m, \quad (15)$$

is positive but not completely positive and decomposable when $\frac{1}{m+1} < y \leq 1$. Especially, if $y = 1$, it reduces to the transpose map; if $y = 0$, it reduces to $\Phi(A) = \text{tr}(A)I$; if $y = \frac{1}{1-m}$, it reduces to $\Phi(A) = \text{tr}(A)I - A^t$. One can check that $\Phi_4^{m,y}$ is completely positive iff $-\frac{1}{m-1} \leq y \leq \frac{1}{m+1}$ and is positive iff $-\frac{1}{m-1} \leq y \leq 1$.

Remark The maps Φ_1^a , Φ_2^a , $\Phi_3^{m,x}$ and $\Phi_4^{m,y}$ range from positive but not completely positive ones to completely positive ones continuously when the parameter a , x and y vary continuously.

4. PNCP maps derived from Hermitian block matrices

We now consider the relation between the positive maps and the Hermitian block matrices. Let $A \in \mathcal{M}_m \otimes \mathcal{M}_n$, then A can be denoted by both $[A_{ij}]$ and $[\tilde{A}_{kl}]$ as in Eq. (10). For any unitary matrix $U \in \mathcal{M}_m$, let $\tilde{A}_{kl}^U = U \tilde{A}_{kl} U^\dagger$. In the following, we write

$$A^U = \sum_{i,j=1}^m E_{ij} \otimes A_{ij}^U = \sum_{k,l=1}^n \tilde{A}_{kl}^U \otimes E_{kl}, \quad (16)$$

that is $A^U = (U \otimes I_n)A(U^\dagger \otimes I_n) = [\tilde{A}_{kl}^U] = [A_{ij}^U]$.

Theorem 4.1. *Let A be a Hermitian matrix in $\mathcal{M}_m \otimes \mathcal{M}_n$ and $A \not\geq 0$. If $A_{ii} \geq 0$ and $A_{11}^U \geq 0$ for any unitary matrix U , then $\Psi(E_{ij}) = A_{ij}$ is a PNCP map.*

Proof If $A_{ii} \geq 0$ and $A_{11}^U \geq 0$ for any unitary matrix U , then for any rank-one projection $|w\rangle\langle w| \in \mathcal{M}_m$, writing

$$|w\rangle\langle w| = \sum_{i,j=1}^m t_{ij} E_{ij},$$

we have

$$\Psi(|w\rangle\langle w|) = \sum_{i,j=1}^m t_{ij} \Psi(E_{ij}) = \sum_{i,j=1}^m t_{ij} A_{ij} = [\text{tr}(T^t \tilde{A}_{ij})] \geq 0.$$

The inequality above holds since

$$[\text{tr}(T^t \tilde{A}_{ij})] = A_{11}^U$$

for some unitary matrix U . That is, Ψ is positive. On the other hand, $A = (\text{id}_m \otimes \Psi)(\sum_{i,j=1}^m E_{ij} \otimes E_{ij}) \not\geq 0$, thus, it is not completely positive. \square

Then, whether or not such a matrix A exists so that it can lead to a PNCP map? We conjecture that such matrix exists. This is an interesting task and so we will make a further research in the future.

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